

LieP Ring

Database and algorithms for Lie p-rings

2.9.3

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Abstract

LieP Ring gives access to the database of Lie p -rings of order at most p^7 as determined by Mike Newman, Eamonn O'Brien and Michael Vaughan-Lee, see [NOVL03] and [OVL05], and it provides some functionality to work with these Lie p -rings.

If you use LieP Ring, then please cite it as: *Bettina Eick and Michael Vaughan-Lee, LieP Ring -- A GAP Package for computing with nilpotent Lie rings of prime-power order (2014)*, see <https://www.gap-system.org/Packages/liepring.html>

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Acknowledgements

The Lazard correspondence induces a one-to-one correspondence between the Lie p -rings of order p^n and class less than p and the p -groups of order p^n and class less than p . LieP Ring provides a function to evaluate this correspondence; this function has been implemented and given to us by Willem de Graaf.

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Chapter 1

Lie p -rings

In this preliminary chapter we recall some of the theoretic background of Lie rings and Lie p -rings. We refer to Chapter 5 in [Khu98] for some further details. Throughout we assume that p stands for a rational prime.

A Lie ring L is an additive abelian group with a multiplication that is alternating, bilinear and satisfies the Jacobi identity. We denote the product of two elements g and h of L with gh .

A subset $I \subseteq L$ is an *ideal* in the Lie ring L if it is a subgroup of the additive group of L and it satisfies $al \in I$ for all $a \in I$ and $l \in L$. As the multiplication in L is alternating, it follows that $la \in I$ for all $l \in L$ and $a \in I$. Note that if I and J are ideals in L , then $I + J = \{a + b \mid a \in I, b \in J\}$ and $IJ = \langle ab \mid a \in I, b \in J \rangle_+$ are ideals in L .

A subset $U \subseteq L$ is a *subring* of the Lie ring L if U is a Lie ring with respect to the addition and the multiplication of L . Every ideal in L is also a subring of L . As usual, for an ideal I in L the quotient L/I has the structure of a Lie ring, but this does not hold for subrings.

The *lower central series* of the Lie ring L is the series of ideals $L = \gamma_1(L) \geq \gamma_2(L) \geq \dots$ defined by $\gamma_i(L) = \gamma_{i-1}(L)L$. We say that L is *nilpotent* if there exists a natural number c with $\gamma_{c+1}(L) = \{0\}$. The smallest natural number with this property is the *class* of L .

The notion of nilpotence now allows us to state the central definition of this package. A *Lie p -ring* is a Lie ring that is nilpotent and has p^n elements for some natural number n .

Every finite dimensional Lie algebra over a field with p elements is an example of a Lie ring with p^n elements. Note that there exist non-nilpotent Lie algebras of this type: the Lie algebra consisting of all $n \times n$ matrices with trace 0 and $n \geq 3$ is an example. Thus not every Lie ring with p^n elements is nilpotent. (In contrast to the group case, where every group with p^n elements is nilpotent!)

For a Lie p -ring L we define the series $L = \lambda_1(L) \geq \lambda_2(L) \geq \dots$ via $\lambda_{i+1}(L) = \lambda_i(L)L + p\lambda_i(L)$. This series is the *lower exponent- p central series* of L . Its length is the *p -class* of L . If $|L/\lambda_2(L)| = p^d$, then d is the *minimal generator number* of L . Similar to the p -group case, one can observe that this is indeed the cardinality of a generating set of smallest possible size.

Each Lie p -ring L has a central series $L = L_1 \geq \dots \geq L_n \geq \{0\}$ with quotients of order p . Choose $l_i \in L_i \setminus L_{i+1}$ for $1 \leq i \leq n$. Then $(l_1, \dots, l_n)$ is a generating set of L satisfying that $pl_i \in L_{i+1}$ and $l_i l_j \in L_{i+1}$ for $1 \leq j < i \leq n$. We call such a generating sequence a *basis* for L and we say that L has *dimension n* .

Chapter 2

LiePRings in GAP

This package introduces a new datastructure that allows one to define and compute with Lie p -rings in GAP. We first describe this datastructure in the case of ordinary Lie p -rings; that is, Lie p -rings for a fixed prime p with given structure constants. Then we show how this datastructure can also be used to define so-called 'generic' Lie p -rings; that is, Lie p -rings with indeterminate prime p .

2.1 Ordinary Lie p -rings

Let p be a prime and let L be a Lie p -ring of order p^n . Let $(l_1, \dots, l_n)$ be a basis for L . Then there exist coefficients $c_{i,j,k} \in \{0, \dots, p-1\}$ so that the following relations hold in L for $1 \leq i, j \leq n$ with $i \neq j$:

$$l_i \cdot l_j = \sum_{k=i+1}^n c_{i,j,k} l_k,$$

$$p l_i = \sum_{k=i+1}^n c_{i,i,k} l_k.$$

These structure constants define the Lie p -ring L . As the multiplication in a Lie p -ring is anticommutative, it follows that $c_{i,j,k} = -c_{j,i,k}$ holds for each k and each $i \neq j$. Thus the structure constants $c_{i,j,k}$ for $i \geq j$ are sufficient to define the Lie p -ring L .

This package contains the new datastructure *LiePRing* that allows one to define Lie p -rings via their structure constants $c_{i,j,k}$. To use this datastructure, we first collect all relevant information into a record as follows:

dim the dimension n of L ;

prime

the prime p of L ;

tab a list with structure constants $[c_{1,1}, c_{2,1}, c_{2,2}, c_{3,1}, c_{3,2}, c_{3,3}, \dots]$.

Each entry $c_{i,j}$ in the list *tab* is a list $[k_1, c_{i,j,k_1}, k_2, c_{i,j,k_2}, \dots]$ so that $k_1 < k_2 < \dots$ and the entries $c_{i,j,k_1}, c_{i,j,k_2}, \dots$ are the non-zero structure constants in the product $l_i \cdot l_j$. Thus if $l_i \cdot l_j = 0$, then $c_{i,j}$ is the empty list. If an entry in the list *tab* is not bound, then it is assumed to be the empty list.

2.1.1 LiePRingBySCTable

- ▷ LiePRingBySCTable(SC) (function)
 ▷ LiePRingBySCTableNC(SC) (function)

These functions create a *LiePRing* from the structure constants table record *SC*. The first version checks that the multiplication defined by *tab* is alternating and satisfies the Jacobi identity, the second version assumes that this is the case and omits these checks. These checks can also be carried out independently via the following function.

2.1.2 CheckIsLiePRing

- ▷ CheckIsLiePRing(L) (function)

This function takes as input an object *L* created via *LiePRingBySCTableNC* and checks that the Jacobi identity holds in this ring.

The following example creates the Lie 2-ring of order 8 with trivial multiplication.

Example

```

gap> SC := rec( dim := 3, prime := 2, tab := [] );;
gap> L := LiePRingBySCTable(SC);
<LiePRing of dimension 3 over prime 2>
gap> l := BasisOfLiePRing(L);
[ 11, 12, 13 ]
gap> l[1]*l[2];
0
gap> 2*l[1];
0
gap> l[1] + l[2];
11 + 12

```

The next example creates a LiePRing of order 5^4 with non-trivial multiplication.

Example

```

gap> SC := rec( dim := 4, prime := 5, tab := [ [], [3, 1], [], [4, 1] ] );;
gap> L := LiePRingBySCTableNC(SC);;
gap> ViewPCPresentation(L);
[12,11] = 13
[13,11] = 14

```

2.2 Generic Lie p -rings

In a generic Lie p -ring, p is allowed to be an indeterminate and the structure constants are allowed to be rational functions over a polynomial ring in a finite set of commuting indeterminates. It is generally assumed that the indeterminate with name p represents the prime, the indeterminate with name w represents the smallest primitive root modulo the prime and there are further predefined indeterminates with the names $x, y, z, t, j, k, m, n, r, s, u$ and v . These indeterminates are used in the database of Lie p -rings and they can be obtained via

2.2.1 IndeterminateByName

▷ `IndeterminateByName(string)`

(function)

returns the indeterminate with the name *string*.

The structure constants records for generic Lie p -rings are similar to those for ordinary Lie p -rings, but have the additional entry *param* which is a list containing all indeterminates used in the considered Lie p -ring. We exhibit an example.

Example

```
gap> p := IndeterminateByName("p");;
gap> x := IndeterminateByName("x");;
gap> S := rec( dim := 5,
>             param := [ x ],
>             prime := p,
>             tab := [ [ 4, 1 ], [ 3, 1 ], [ 5, x ], [ 4, 1 ], [ 5, 1 ] ] );;
gap> L := LiePRingBySCTable(S);
<LiePRing of dimension 5 over prime p with parameters [ x ]>
gap> ViewPCPresentation(L);
p*11 = 14
p*12 = x*15
[12,11] = 13
[13,11] = 14
[13,12] = 15
gap> l := BasisOfLiePRing(L);
[ 11, 12, 13, 14, 15 ]
gap> p*l[1];
14
gap> l[1]+l[2];
11 + 12
gap> l[1]*l[2];
-1*13
```

2.3 Specialising Lie p -rings

A generic Lie p -ring defines a family of ordinary Lie p -rings by evaluating the parameters contained in its presentation. It is generally assumed that the indeterminate p is evaluated to a rational prime P and the indeterminate w is evaluated to the smallest primitive root modulo P (this can be determined via `PrimitiveRootMod(P)`). All other indeterminates can take arbitrary integer values (usually these values are in $\{0, \dots, P-1\}$, but other choices are possible as well). The following functions allow one to evaluate the indeterminates.

2.3.1 SpecialiseLiePRing

▷ `SpecialiseLiePRing(L, P, para, vals)`

(function)

takes as input a generic Lie p -ring L , a rational prime P , a list of indeterminates *para* and a corresponding list of values *vals*. The function returns a new Lie p -ring in which the prime p is evaluated to P , the parameter w is evaluated to `PrimitiveRootMod(P)` and the parameters in *para* are evaluated to *vals*.

2.3.2 SpecialisePrimeOfLieP Ring

▷ `SpecialisePrimeOfLieP Ring(L, P)`

(function)

this is a shortcut for `SpecialiseLieP Ring(L, P, [], [])`. We exhibit some example applications.

Example

```
gap> p := IndeterminateByName("p");;
gap> w := IndeterminateByName("w");;
gap> x := IndeterminateByName("x");;
gap> y := IndeterminateByName("y");;
gap> S := rec( dim := 7,
>             param := [ w, x, y ],
>             prime := p,
>             tab := [ [ ], [ 6, 1 ], [ 6, 1 ], [ 7, 1 ], [ ],
>                     [ 6, x, 7, y ], [ ], [ 7, 1 ], [ 6, w ] ] );;
gap> L := LieP RingBySCTable(S);
<LieP Ring of dimension 7 over prime p with parameters [ w, x, y ]>
gap> ViewPCPresentation(L);
p*12 = 16
p*13 = x*16 + y*17
[12,11] = 16
[13,11] = 17
[14,12] = 17
[14,13] = w*16
gap> SpecialiseLieP Ring(L, 7, [x, y], [0,0]);
<LieP Ring of dimension 7 over prime 7>
gap> ViewPCPresentation(last);
7*12 = 16
[12,11] = 16
[13,11] = 17
[14,12] = 17
[14,13] = 3*16
gap> SpecialiseLieP Ring(L, 11, [x, y], [0,10]);
<LieP Ring of dimension 7 over prime 11>
gap> ViewPCPresentation(last);
11*12 = 16
11*13 = 10*17
[12,11] = 16
[13,11] = 17
[14,12] = 17
[14,13] = 2*16
gap> Cartesian([0,1],[0,1]);
[ [ 0, 0 ], [ 0, 1 ], [ 1, 0 ], [ 1, 1 ] ]
gap> List(last, v -> SpecialiseLieP Ring(L, 2, [x,y], v));
[ <LieP Ring of dimension 7 over prime 2>,
  <LieP Ring of dimension 7 over prime 2>,
  <LieP Ring of dimension 7 over prime 2>,
  <LieP Ring of dimension 7 over prime 2> ]
```

It is not necessary to specialise all parameters at once. In particular, it is possible to leave the prime p as indeterminate and specialise only some of the parameters. (Except for w which is linked to p .)

Example

```
gap> SpecialiseLieP Ring(L, p, [x], [0]);
<LieP Ring of dimension 7 over prime p with parameters [ y, w ]>
gap> ViewPCPresentation(last);
p*12 = 16
p*13 = y*17
[12,11] = 16
[13,11] = 17
[14,12] = 17
[14,13] = w*16
gap> SpecialiseLieP Ring(L, p, [y], [3]);
<LieP Ring of dimension 7 over prime p with parameters [ x, w ]>
gap> ViewPCPresentation(last);
p*12 = 16
p*13 = x*16 + 3*17
[12,11] = 16
[13,11] = 17
[14,12] = 17
[14,13] = w*16
```

It is also possible to specialise the prime only, but leave all or some of the parameters indeterminate. Note that specialising p also specialises w . Again, we continue to use the generic Lie p -ring L as above.

Example

```
gap> SpecialisePrimeOfLieP Ring(L, 29);
<LieP Ring of dimension 7 over prime 29 with parameters [ y, x ]>
gap> ViewPCPresentation(last);
29*12 = 16
29*13 = x*16 + y*17
[12,11] = 16
[13,11] = 17
[14,12] = 17
[14,13] = 2*16
```

2.3.3 LiePValues

▷ LiePValues(K)

(attribute)

if K is obtained by specialising, then this attribute is set and contains the parameters that have been specialised and their values.

Example

```
gap> L := LiePRingsByLibrary(6)[14];
<LieP Ring of dimension 6 over prime p with parameters [ x ]>
gap> K := SpecialisePrimeOfLieP Ring(L, 5);
<LieP Ring of dimension 6 over prime 5 with parameters [ x ]>
gap> LiePValues(K);
[ [ p, w ], [ 5, 2 ] ]
```

2.4 Subrings of Lie p -rings

Let L be a Lie p -ring with basis $(l_1, \dots, l_n)$ and let U be a subring of L . Then U is a Lie p -ring and thus also has a basis $(u_1, \dots, u_m)$. For $1 \leq i \leq m$ we define the coefficients $a_{i,j} \in \{0, \dots, p-1\}$ via

$$u_i = \sum_{j=1}^n a_{i,j} l_j$$

and we denote with A the matrix with entries $a_{i,j}$. We say that the basis $(u_1, \dots, u_m)$ is *induced* if A is in upper triangular form. Further, the basis $(u_1, \dots, u_m)$ is *canonical* if A is in upper echelon form; that is, it is upper triangular, each row in A has leading entry 1 and there are 0's above the leading entry. Note that a canonical basis is unique for the subring.

2.4.1 LiePSubring

▷ LiePSubring(L , $gens$)

(function)

Let L be a (generic or ordinary) Lie p -ring and let $gens$ be a set of elements in L . This function determines a canonical basis for the subring generated by $gens$ in L and returns the LiePSubring of L generated by $gens$. Note that this function may have strange effects for generic Lie p -rings as the following example shows.

Example

```
gap> L := LiePRingsByLibrary(6)[100];
<LiePRing of dimension 6 over prime p>
gap> l := BasisOfLiePRing(L);
[ 11, 12, 13, 14, 15, 16 ]
gap> U := LiePSubring(L, [5*l[1]]);
<LiePRing of dimension 3 over prime p>
gap> BasisOfLiePRing(U);
[ 11, 14, 16 ]
gap> K := SpecialisePrimeOfLiePRing(L, 5);
<LiePRing of dimension 6 over prime 5>
gap> b := BasisOfLiePRing(K);
[ 11, 12, 13, 14, 15, 16 ]
gap> LiePSubring(K, [5*b[1]]);
<LiePRing of dimension 2 over prime 5>
gap> BasisOfLiePRing(last);
[ 14, 16 ]
gap> K := SpecialisePrimeOfLiePRing(L, 7);
<LiePRing of dimension 6 over prime 7>
gap> b := BasisOfLiePRing(K);
[ 11, 12, 13, 14, 15, 16 ]
gap> U := LiePSubring(K, [5*b[1]]);
<LiePRing of dimension 3 over prime 7>
gap> BasisOfLiePRing(U);
[ 11, 14, 16 ]
```

2.4.2 LiePIdeal

▷ LiePIdeal(L , $gens$)

(function)

return the ideal of L generated by *gens*. This function computes an induced basis for the ideal.

Example

```
gap> LiePIdeal(L, [l[1]]);
<LieP Ring of dimension 5 over prime p>
gap> BasisOfLieP Ring(last);
[ 11, 13, 14, 15, 16 ]
```

2.4.3 LiePQuotient

▷ LiePQuotient(L, U) (function)

return a Lie p -ring isomorphic to L/U where U must be an ideal of L . This function requires that L is an ordinary Lie p -ring.

Example

```
gap> LiePIdeal(K, [b[1]]);
<LieP Ring of dimension 5 over prime 7>
gap> LiePIdeal(K, [b[2]]);
<LieP Ring of dimension 4 over prime 7>
gap> LiePQuotient(K,last);
<LieP Ring of dimension 2 over prime 7>
```

2.5 Elementary functions

The functions described in this section work for ordinary and generic Lie p -rings and their subrings.

2.5.1 PrimeOfLieP Ring

▷ PrimeOfLieP Ring(L) (attribute)

returns the underlying prime. This can either be an integer or an indeterminate.

2.5.2 BasisOfLieP Ring

▷ BasisOfLieP Ring(L) (attribute)

returns a basis for L .

2.5.3 DimensionOfLieP Ring

▷ DimensionOfLieP Ring(L) (attribute)

returns the dimension of L .

2.5.4 ParametersOfLieP Ring

▷ ParametersOfLieP Ring(L) (function)

returns the list of indeterminates involved in L . If L is a subring of a Lie p -ring defined by structure constants, then the parameters of the parent are returned.

2.5.5 ViewPCPresentation

▷ ViewPCPresentation(L) (function)

prints the presentation for L with respect to its basis.

2.6 Series of subrings

Let L be a generic or ordinary Lie p -ring or a subring of such a Lie p -ring.

2.6.1 LiePLowerCentralSeries

▷ LiePLowerCentralSeries(L) (function)

returns the lower central series of L .

2.6.2 LiePLowerPCentralSeries

▷ LiePLowerPCentralSeries(L) (function)

returns the lower exponent- p central series of L .

2.6.3 LiePDerivedSeries

▷ LiePDerivedSeries(L) (function)

returns the derived series of L .

2.6.4 LiePMinimalGeneratingSet

▷ LiePMinimalGeneratingSet(L) (function)

returns a minimal generating set of L ; that is, a generating set of smallest possible size.

2.7 The Lazard correspondence

The following function has been implemented by Willem de Graaf. It uses the Baker-Campbell-Hausdorff formula as described in [CdGVL12] and it is based on the Liering package [CdG10].

2.7.1 PGroupByLieP Ring

▷ PGroupByLieP Ring(L) (function)

Let L be an ordinary Lie p -ring with $cl(L) < p$. Then this function returns the p -group G obtained from L via the Lazard correspondence.

Chapter 3

The Database

This package gives access to the database of Lie p -rings of order at most p^7 as determined by Mike Newman, Eamonn O'Brien and Michael Vaughan-Lee, see [NOVL03] and [OVL05]. A description of the database can also be found in [VL13].

For each $n \in \{1, \dots, 7\}$ this package contains a (finite) list of generic presentations of Lie p -rings. For each prime $p \geq 5$, each of the generic Lie p -rings gives rise to a family of Lie p -rings over the considered prime p by specialising the indeterminates to a certain list of values. The resulting lists of Lie p -rings provide a complete and irredundant set of isomorphism type representatives of the Lie p -rings of order p^n . The generic Lie p -rings of p -class at most 2 can also be considered for the prime $p = 3$ and yield a list of isomorphism type representatives for the Lie p -rings of order 3^n and p -class at most 2.

The Lazard correspondence has been used to check the correctness of the database of Lie p -rings: for various small primes it has been checked that the Lie p -rings of this database define non-isomorphic finite p -groups.

In the following we describe functions to access the database. Throughout this chapter, we assume that $\dim \in \{1, \dots, 7\}$ and P is a prime with $P \neq 2$.

3.1 Accessing Lie p -rings

3.1.1 LiePRingsByLibrary

▷ `LiePRingsByLibrary(dim[, gen][, cl])` (function)

returns the generic Lie p -rings of dimension $\dim$ in the database. The second form returns the Lie p -rings of minimal generator number gen and p -class cl only.

3.1.2 LiePRingsByLibrary (for a prime)

▷ `LiePRingsByLibrary(dim, P[, gen][, cl])` (function)

returns isomorphism type representatives of ordinary Lie p -rings of dimension $\dim$ for the prime P . The second form returns the Lie p -rings of minimal generator number gen and p -class cl only. The function assumes $P \geq 3$ and for $P = 3$ there are only the Lie p -rings of p -class at most 2 available.

The first example yields the generic Lie p -rings of dimension 4.

Example

```
gap> LiePRingsByLibrary(4);
[ <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p> ]
```

The next example yields the isomorphism type representatives of Lie p -rings of dimension 3 for the prime 5.

Example

```
gap> LiePRingsByLibrary(3, 5);
[ <LiePRing of dimension 3 over prime 5>,
  <LiePRing of dimension 3 over prime 5>,
  <LiePRing of dimension 3 over prime 5>,
  <LiePRing of dimension 3 over prime 5>,
  <LiePRing of dimension 3 over prime 5> ]
```

The following example extracts the generic Lie p -rings of dimension 5 with minimal generator number 2 and p -class 4.

Example

```
gap> LiePRingsByLibrary(5, 2, 4);
[ <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p> ]
```

Finally, we determine the isomorphism type representatives of Lie p -rings of dimension 5, minimal generator number 2 and p -class 4 for the prime 7.

Example

```
gap> LiePRingsByLibrary(5, 7, 2, 4);
[ <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7> ]
```

3.2 Numbers of Lie p -rings

3.2.1 NumberOfLiePRings

▷ `NumberOfLiePRings(dim)` (function)

returns the number of generic Lie p -rings in the database of the considered dimension for $\dim \in \{1, \dots, 7\}$.

Example

```
gap> List([1..7], x -> NumberOfLiePRings(x));
[ 1, 2, 5, 15, 75, 542, 4773 ]
```

3.2.2 NumberOfLiePRings (for a prime)

▷ `NumberOfLiePRings(dim, P)` (function)

returns the number of isomorphism types of ordinary Lie p -rings of order $P^{\dim}$ in the database. If $P \geq 5$, then this is the number of all isomorphism types of Lie p -rings of order $P^{\dim}$ and if $P = 3$ then this is the number of all isomorphism types of Lie p -rings of p -class at most 2. If $P \geq 7$, then this number coincides with `NumberSmallGroups( $P^{\dim}$ )`.

3.2.3 NumberOfLiePRingsInFamily

▷ `NumberOfLiePRingsInFamily(L)` (function)

returns the number of Lie p -rings associated to L as a polynomial in p and possibly some residue classes.

Example

```
gap> L := LiePRingsByLibrary(7)[780];
<LiePRing of dimension 7 over prime p with parameters
[ x, y, z, t, s, u, v ]>
gap> NumberOfLiePRingsInFamily(L);
```

$$-1/3*p^5*(p-1,3)+p^5-1/3*p^4*(p-1,3)+p^4-1/3*p^3*(p-1,3)+p^3-1/3*p^2*(p-1,3)+p^2-p*(p-1,3)+3*p-3/2*(p-1,3)+9/2$$

3.3 Searching the database

We now consider a generic Lie p -ring L from the database and consider the family of ordinary Lie p -rings that arise from it.

3.3.1 LiePRingsInFamily

▷ LiePRingsInFamily(L , P) (function)

takes as input a generic Lie p -ring L from the database and a prime P and returns all Lie p -rings determined by L and P up to isomorphism. This function returns fail if the generic Lie p -ring does not exist for the special prime P ; this may be due to the conditions on the prime or (if $P = 3$) to the p -class of the Lie p -ring.

Example

```
gap> L := LiePRingsByLibrary(7)[118];
<LiePRing of dimension 7 over prime p with parameters [ x, y ]>
gap> LibraryConditions(L);
[ "[x,y]~[x,-y]", "p=1 mod 4" ]
gap> LiePRingsInFamily(L, 7);
fail
gap> Length(LiePRingsInFamily(L,13));
91
gap> 13^2;
169
```

The following example shows how to determine all Lie p -rings of dimension 5 and p -class 4 over the prime 29 up to isomorphism.

Example

```
gap> L := LiePRingsByLibrary(5);
gap> L := Filtered(L, x -> PClassOfLiePRing(x)=4);
[ <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p> ]
gap> K := List(L, x-> LiePRingsInFamily(x, 29));
[ [ <LiePRing of dimension 5 over prime 29> ],
```

```

[ <LiePRing of dimension 5 over prime 29> ],
[ <LiePRing of dimension 5 over prime 29> ], fail, fail,
[ <LiePRing of dimension 5 over prime 29> ],
[ <LiePRing of dimension 5 over prime 29> ],
[ <LiePRing of dimension 5 over prime 29> ],
[ <LiePRing of dimension 5 over prime 29> ],
[ <LiePRing of dimension 5 over prime 29> ], fail, fail,
[ <LiePRing of dimension 5 over prime 29> ],
[ <LiePRing of dimension 5 over prime 29> ] ]
gap> K := Filtered(Flat(K), x -> x<>fail);
[ <LiePRing of dimension 5 over prime 29>,
  <LiePRing of dimension 5 over prime 29>,
  <LiePRing of dimension 5 over prime 29>,
  <LiePRing of dimension 5 over prime 29>,
  <LiePRing of dimension 5 over prime 29>,
  <LiePRing of dimension 5 over prime 29>,
  <LiePRing of dimension 5 over prime 29>,
  <LiePRing of dimension 5 over prime 29>,
  <LiePRing of dimension 5 over prime 29>,
  <LiePRing of dimension 5 over prime 29>,
  <LiePRing of dimension 5 over prime 29> ]

```

3.4 More details

Let L be a Lie p -ring from the database. Then the following additional attributes are available.

3.4.1 LibraryName

▷ `LibraryName( $L$ )` (attribute)

returns a string with the name of L in the database. See `p567.pdf` for further background.

3.4.2 ShortPresentation

▷ `ShortPresentation( $L$ )` (attribute)

returns a string exhibiting a short presentation of L .

3.4.3 LibraryConditions

▷ `LibraryConditions( $L$ )` (attribute)

returns the conditions on L . This is a list of two strings. The first string exhibits the conditions on the parameters of L , the second shows the conditions on primes.

3.4.4 MinimalGeneratorNumberOfLiePRing

▷ MinimalGeneratorNumberOfLiePRing(L) (attribute)

returns the minimal generator number of L .

3.4.5 PClassOfLiePRing

▷ PClassOfLiePRing(L) (attribute)

returns the p -class of L .

Example

```
gap> L := LiePRingsByLibrary(7)[118];
<LiePRing of dimension 7 over prime p with parameters [ x, y ]>
gap> LibraryName(L);
"7.118"
gap> LibraryConditions(L);
[ "[x,y]~[x,-y]", "p=1 mod 4" ]
```

All of the information listed in this section is inherited when L is specialised.

Example

```
gap> L := LiePRingsByLibrary(7)[118];
<LiePRing of dimension 7 over prime p with parameters [ x, y ]>
gap> K := SpecialiseLiePRing(L, 13, ParametersOfLiePRing(L), [0,0]);
<LiePRing of dimension 7 over prime 13>
gap> LibraryName(K);
"7.118"
gap> LibraryConditions(K);
[ "[x,y]~[x,-y]", "p=1 mod 4" ]
```

The following example shows how to find a Lie p -ring with a given name in the database.

Example

```
gap> L := LiePRingsByLibrary(7);;
gap> Filtered(L, x -> LibraryName(x) = "7.1010")[1];
<LiePRing of dimension 7 over prime p>
```

3.5 Special functions for dimension 7

The database of Lie p -rings of dimension 7 is very large and it may be time-consuming (or even impossible due to storage problems) to generate all Lie p -rings of dimension 7 for a given prime P .

Thus there are some special functions available that can be used to access a particular set of Lie p -rings of dimension 7 only. In particular, it is possible to consider the descendants of a single Lie p -ring of smaller dimension by itself. The Lie p -rings of this type are all stored in one file of the library. Thus, equivalently, it is possible to access the Lie p -rings in one single file only.

The table LIE_TABLE contains a list of all possible files together with the number of Lie p -rings generated by their corresponding Lie p -rings.

3.5.1 LiePRingsDim7ByFile

▷ LiePRingsDim7ByFile(*nr*) (function)

returns the generic Lie p -rings in file number *nr*.

3.5.2 LiePRingsDim7ByFile (for a prime)

▷ LiePRingsDim7ByFile(*nr*, *P*) (function)

returns the isomorphism types of Lie p -rings in file number *nr* for the prime *P*.

Example

```
gap> LIE_TABLE[100];
[ "3gen/gapdec6.139", 1/2*p+(p-1,3)+3/2 ]
gap> LiePRingsDim7ByFile(100);
[ <LiePRing of dimension 7 over prime p>,
  <LiePRing of dimension 7 over prime p>,
  <LiePRing of dimension 7 over prime p>,
  <LiePRing of dimension 7 over prime p>,
  <LiePRing of dimension 7 over prime p with parameters [ x ]> ]
gap> LiePRingsDim7ByFile(100, 7);
[ <LiePRing of dimension 7 over prime 7>,
  <LiePRing of dimension 7 over prime 7>,
  <LiePRing of dimension 7 over prime 7>,
  <LiePRing of dimension 7 over prime 7>,
  <LiePRing of dimension 7 over prime 7>,
  <LiePRing of dimension 7 over prime 7>,
  <LiePRing of dimension 7 over prime 7>,
  <LiePRing of dimension 7 over prime 7> ]
```

3.6 Dimension 8 and maximal class

Recently, Lee and Vaughan-Lee [LVL22] determined the Lie p -rings of dimension 8 with maximal class up to isomorphism. This classification is now also available in the Lie p -ring package via the following functions.

3.6.1 LiePRingsByLibraryMC8

▷ LiePRingsByLibraryMC8() (function)

returns a list of 69 generic Lie p -rings.

3.6.2 LiePRingsInFamilyMC8

▷ LiePRingsInFamilyMC8(*L*, *P*) (function)

returns the isomorphism types of Lie p -rings in the family defined by the generic Lie p -ring *L* for a fixed prime *P* with $P \geq 5$.

Chapter 4

Advanced functions for Lie p -rings

This chapter describes a few more advanced functions available for generic Lie p -rings.

4.1 Schur multipliers

The package contains a method to determine the Schur multipliers of the Lie p -rings in the family defined by a generic Lie p -ring.

4.1.1 LiePSchurMult

▷ LiePSchurMult(L) (function)

The function takes as input a generic Lie p -ring L and determines a list of possible Schur multipliers, each described by its abelian invariants, for the Lie p -rings in the family described by L . For each entry in the list of Schur multipliers there is a description of those parameters which give the considered entry. This description consists of two lists `units` and `zeros`. Both consist of rational functions over the parameters of the Lie p -ring. The parameters described by these lists are those which evaluate to zero for each rational function in `zeros` and do not evaluate to zero for each rational function in `units`.

Example

```
gap> LL := LiePRingsByLibrary(7);
gap> L := Filtered(LL, x -> Length(ParametersOfLiePRing(x))=2)[1];
<LiePRing of dimension 7 over prime p with parameters [ x, y ]>
gap> NumberOfLiePRingsInFamily(L);
p^2-p
gap> RingInvariants(L);
rec( units := [ x ], zeros := [ ] )
gap> ss := LiePSchurMult(L);
[ rec( norm := [ p ], units := [ x, y ], zeros := [ x*y^2-x*y+1 ] ),
  rec( norm := [ p^2 ], units := [ x ], zeros := [ x*y ] ),
  rec( norm := [ p ], units := [ x, x*y^2-x*y+1, y ], zeros := [ ] ) ]
```

In this example, L defines a generic Lie p -ring with two parameters and the `RingInvariants` of L show that the parameter x should be non-zero. The function `LiePSchurMult(L)` yields that there are two possible Schur multipliers for the Lie p -rings in the family defined by L : the cyclic groups of order p and of order p^2 . The second option only arises if $xy = 0$ and thus, as x is non-zero, if $y = 0$.

The package also contains a function that tries to determine the numbers of values of the parameters satisfying the conditions of a description of a Schur multiplier. This succeeds in many cases and returns a polynomial in p in this case. If it does not succeed then it returns `fail`.

4.1.2 ElementNumbers

▷ `ElementNumbers(pp, ss)`

(function)

We continue the above example.

Example

```
gap> ElementNumbers(ParametersOfLiePRing(L), ss);
rec( norms := [ [ p^2 ], [ p ] ], numbs := [ p-1, p^2-2*p+1 ] )
```

4.2 Automorphism groups

The package contains a function that determines a description for the automorphism groups of the Lie p -rings in the family defined by a generic Lie p -ring.

4.2.1 AutGroupDescription

▷ `AutGroupDescription(L)`

(function)

Each automorphism of L is defined by its images on a generating set of L . If $l_1, \dots, l_n$ is a basis of L and $l_1, \dots, l_d$ is a generating set, then each automorphism is defined by the images of $l_1, \dots, l_d$ and each image is an integral linear combination of the basis elements $l_1, \dots, l_n$. The function `AutGroupDescription` returns a matrix containing a description of the coefficients in each linear combination and a list of relations among these coefficients. We consider two examples.

Example

```
gap> L := Filtered(LL, x -> Length(ParametersOfLiePRing(x))=2)[1];
<LiePRing of dimension 7 over prime p with parameters [ x, y ]>
gap> AutGroupDescription(L);
rec( auto := [ [ 1, 0, A13, A14, A15, A16, A17 ],
               [ 0, 1, A23, A24, A25, A26, A27 ] ],
      eqns := [ [ ], [ ] ] )
gap> L := Filtered(LL, x -> Length(ParametersOfLiePRing(x))=2)[2];
<LiePRing of dimension 7 over prime p with parameters [ x, y ]>
gap> AutGroupDescription(L);
rec( auto := [ [ A22^3, 0, A13, A14, A15, A16, A17 ],
               [ 0, A22, A23, A24, A25, A26, A27 ] ],
      eqns := [ [ A22*A24-1/2*A23^2, A22^2*y-y,
                  A22*A23^2*y-2*A24*y, A22^4-1,
                  A23^4*y-4*A24^2*y, A22^3*A23^2-2*A24,
                  A22^2*A23^4-4*A24^2, A22*A23^6-8*A24^3,
                  A23^8-16*A24^4 ] ] ] )
```

In both cases, L is generated by the first two entries in its basis and hence the automorphism group matrix has two rows and seven columns. In the first case, L has p^{10} automorphisms inducing the identity on the Frattini-quotient of L . In the second case, the automorphism group matrix shows that each automorphism induces a certain type of diagonal matrix on the Frattini-quotient of L and there

are further equations among the coefficients of the matrix. These further equations are equivalent to $A_{22}^2 = 1$ and $A_{24} = A_{22}A_{23}^2/2$. Hence L has $2p^9$ automorphisms.

The entry `eqns` is a list of lists. The equations in the i -th entry of this list have to be satisfied mod p^i .

In a few special cases, the function returns a list of possible automorphisms together with related equations and conditions. We exhibit an example.

Example

```
gap> L := LiePRingsByLibrary(7)[489];
<LiePRing of dimension 7 over prime p with parameters [ x ]>
gap> AutGroupDescription(L);
[ rec( auto := [ [ 1, 0, A13, A14, A15, A16, A17 ],
                  [ 0, 1, A23, A24, A25, A26, A27 ] ],
  comment := "p^8 automorphisms",
  eqns := [ [ A13^2*x-A13*A23+2*A15*x+A14-A25,
              -A13*A23*x+A14*x+A23^2-A25*x-2*A24 ] ] ),
  rec( auto := [ [ 0, A12, A13, A14, A15, A16, A17 ],
                  [ -x, 0, A23, A24, A25, A26, A27 ] ],
  comment := "p^8 automorphisms when x <> 0 mod p",
  eqns := [ [ A12^2*A24*x-A12*A13*A23*x+A12*A13*x^2
              +2*A12*A15*x^2+A12*A14*x-A13^2*x+A13*x+A15*x-A14,
              -A12^2*A23*x^3+A12*A13*x^3+A12*A23^2*x-A12*A25*x^2
              -2*A12*A24*x+A13*A23*x+A13*x^2-A15*x^2+A23*x+A25*x-A24 ],
              [ A12*x+1 ] ] ) ]
```

In this example $A_{12}x = -1$ modulo p^2 . We note that different choices for A_{12} do not give different automorphisms. Hence a single solution for A_{12} is sufficient to describe all automorphisms.

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